Compactness and distance to spaces of continuous functions

B. Cascales

Universidad de Murcia

Palermo, Italy. June 9 - 16, 2007

The papers

- B. Cascales, W. Marciszesky, and M. Raja, *Distance to spaces of continuous functions*, Topology Appl. **153** (2006), 2303–2319.
- C. Angosto and B. Cascales, *The quantitative difference between countable compactness and compactness*, Submitted, 2006.
- C. Angosto, B. Cascales and I. Namioka, *Distances to spaces of Baire one functions*, Submitted, 2007.

1. The starting point. . . our goals

2. The results
   - $C(K)$ spaces: a taste for simple things
   - Applications to Banach spaces
   - Results for $C(X)$ and $B_1(X)$ spaces

3. References

The starting point...

List | Details

- M. Fabian, P. Hájek, V. Montesinos, and V. Zizler.
  A quantitative version of Krein's Theorem..
  Rev. Mat. Iberoamericana **21** (2005), no. 1, 237–248..
- A. S. Granero.
  An extension of Krein-Šmulian theorem.
  Rev. Mat. Iberoamericana **22** (2005), no. 1, 93–110.
- A. S. Granero, P. Hájek, and V. Montesinos Santalucía.
  Convexity and w^*-compactness in Banach spaces.
  Math. Ann., 2005.

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Let E be a Banach space and let $H \subset E$ be a bounded subset of E. Then

$$\widehat{\mathrm{d}}(\overline{\mathrm{co}(H)}, E) \leq 2\widehat{\mathrm{d}}(\overline{H}, E),$$

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- closures are weak*-closures taken in the bidual E^{**};
- $\widehat{\mathrm{d}}(A, E) := \sup\{d(a, E) : a \in A\}$ for $A \subset E^{**}$;
- $\widehat{\mathrm{d}}(A, E) = 0$ iff $A \subset E$. Hence the inequality implies Krein's theorem (if H is relatively weakly compact then $\overline{\mathrm{co}(H)}$ is weakly compact.)

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- Some of the constant involved are sharp.

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- To take the results where *(I think!)* they belong *i.e.* to the context of $C(K)$ and $\mathbb{R}^K$ spaces endowed with τ_p;

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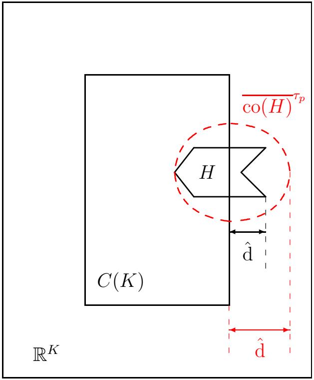

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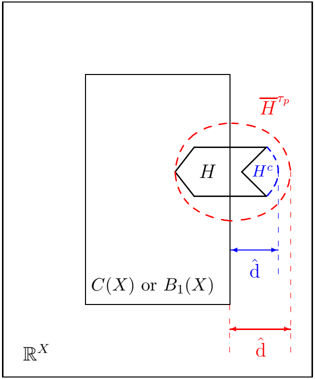

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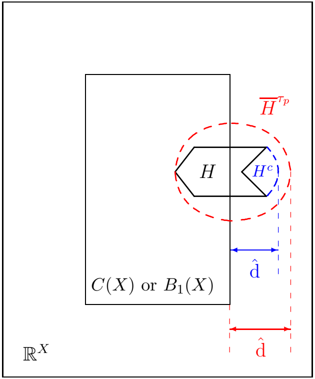

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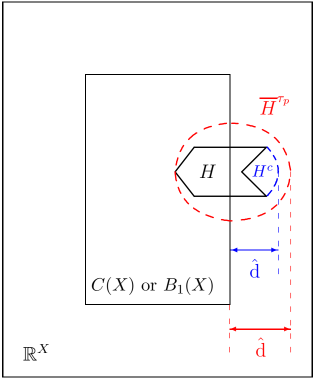

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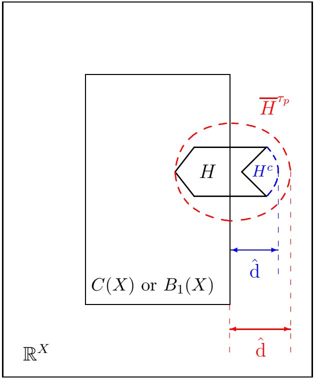

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tools

- new reading of the *classical*;
- for $C(X)$ we use *double limits* used by Grothendieck;
- for $B_1(X)$ we use the notions of *fragmentability* and *σ-fragmentability* of functions.

Quantitative Grothendieck charact. of τ_p-compactness

Theorem

If K is a compact topological space and H is a uniformly bounded subset of $C(K)$, then

$$\mathrm{ck}(H) \leq \hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \gamma(H) \leq 2\,\mathrm{ck}(H).$$

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If H is relatively countably compact in $C(K)$ then $\mathrm{ck}(H) = 0$

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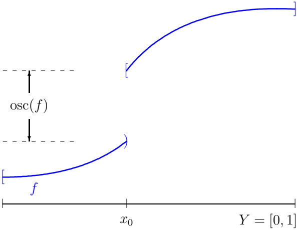

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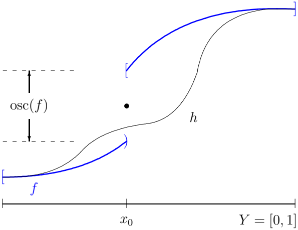

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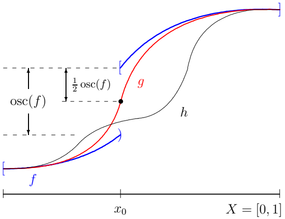

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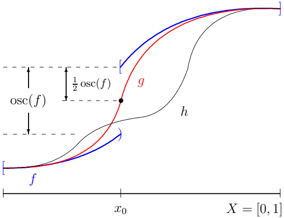
- In particular $\mathrm{osc}(f,x) \leq 2\gamma(H)$ for every $x \in K$;
- $d(f, C(K))) = \frac{1}{2} \sup_{x \in K} \mathrm{osc}(f,x) \leq \gamma(H)$.

Theorem

If K is a compact topological space and H is a uniformly bounded subset of $C(K)$, then

$$\mathrm{ck}(H) \overset{(a)}{\leq} \hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)) \overset{(b)}{\leq} \gamma(H) \overset{(c)}{\leq} 2\,\mathrm{ck}(H).$$



(a) is obvious.

(b)

- in $\gamma(H)$ replace sequences by nets.
- Pick $f \in \overline{H}^{\mathbb{R}^K}$ and fix $x \in K$.
- Take a net $(x_\alpha) \to x$ in K such that

$$\lim_\alpha |f(x_\alpha) - f(x)| = \inf_U \sup_{y \in U} |f(y) - f(x)| =: \mathrm{osc}^*(f,x);$$

- Take a net in H $(f_\beta) \to f$ in $\mathbb{R}^K$.
- Assume (we can!) $f(x_\alpha) \to z$ in $\mathbb{R}$
- We get

$$\lim_\alpha \lim_\beta f_\beta(x_\alpha) = \lim_\alpha f(x_\alpha) = z$$

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If K is a compact topological space and H be a uniformly bounded subset and a uniformly bounded subset H of $\mathbb{R}^K$ we have that

$$\gamma(H) = \gamma(\mathrm{co}(H)),$$

and as a consequence we obtain for $H \subset C(K)$ that

$$\hat{\mathrm{d}}(\overline{\mathrm{co}(H)}^{\mathbb{R}^K}, C(K)) \leq 2\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{1}$$

and in the general case $H \subset \mathbb{R}^K$

$$\hat{\mathrm{d}}(\overline{\mathrm{co}(H)}^{\mathbb{R}^K}, C(K)) \leq 5\hat{\mathrm{d}}(\overline{H}^{\mathbb{R}^K}, C(K)). \tag{2}$$

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2. When $H \subset \mathbb{R}^K$, we approximate H by some set in $C(K)$, then use (1) and 5 appears as a simple

$$5 = 2 \times 2 + 1.$$

Distances to spaces of affine continuous functions

Theorem

If K is compact convex subset of a l.c.s. and $f \in \mathscr{A}(K)$ then

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$$\delta > \operatorname{osc}(f) \geq \inf_{U \in \mathscr{U}_x} \sup_{y,z \in U} (f(y) - f(z))$$

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$$f_2(x) := \sup_{U \in \mathscr{U}_x} \inf_{z \in U} f(z) + \frac{\delta}{2}$$

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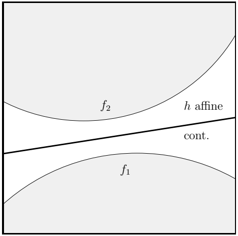

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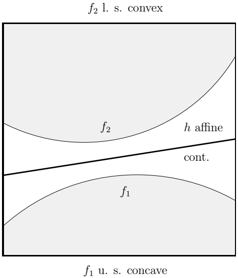

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4. Squeeze h between f_2 and f_1 and $\|f - h\|_\infty \leq \delta/2$.

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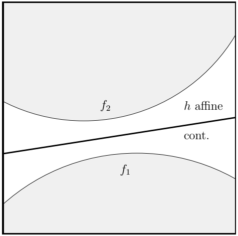

f_1 u. s. concave

Corollary

Let X be a Banach space and let B_{X^*} be the closed unit ball in the dual X^* endowed with the w^*-topology. Let $i: X \to X^{**}$ and $j: X^{**} \to \ell_\infty(B_{X^*})$ be the canonical embedding. Then, for every $x^{**} \in X^{**}$ we have:

$$d(x^{**}, i(X)) = d(j(x^{**}), C(B_{X^*})).$$

Measures of weak noncompactness

Definition

Given a bounded subset H of a Banach space E we define:

$$\gamma(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset B_{E^*}, (x_n) \subset H\},$$

assuming the involved limits exist,

$$\mathrm{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{w^*}, E\Big),$$

$$\mathrm{k}(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup_{x^{**}\in \overline{H}^{w^*}} d(x^{**}, E),$$

where the w^*-closures are taken in E^{**} and the distance d is the usual inf distance for sets associated to the natural norm in E^{**}.

Relationship between measures of weak noncompactness

Theorem

For any bounded subset H of a Banach space E we have:

$$\mathrm{ck}(H) \le \mathrm{k}(H) \le \gamma(H) \le 2\,\mathrm{ck}(H) \le 2\,\mathrm{k}(H)$$

$$\gamma(H) = \gamma(\mathrm{co}(H))$$

For any $x^{**} \in \overline{H}^{w^*}$, there is a sequence $(x_n)_n$ in H such that

$$\|x^{**} - y^{**}\| \le \gamma(H)$$

for any cluster point y^{**} of $(x_n)_n$ in E^{**}. Furthermore, H is weakly relatively compact in E if, and only if, it is zero one (equivalently all) of the numbers $\mathrm{ck}(H), \mathrm{k}(H), \gamma(H)$

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$$\omega(H) := \inf\{\varepsilon > 0 : H \subset K_\varepsilon + \varepsilon B_E \text{ and } K_\varepsilon \subset X \text{ is } w\text{-compact}\},$$

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The result above is the quantitative version of Eberlein-Smulyan and Krein-Smulyan theorems. From $\mathrm{k}(\mathrm{co}(H)) \le 2\,\mathrm{k}(H)$ straightforwardly follows Krein-smulyan theorem.

Other applications to Banach spaces

Theorem (Grothendieck)

Let K be a compact space and let H be a uniformly bounded subset of $C(K)$. Let us define

$$\gamma_K(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset H, (x_n) \subset K\},$$

assuming the involved limits exist. Then we have

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Theorem (Gantmacher)

Let E and F be Banach spaces, $T : E \to F$ an operator and $T^* : F^* \to E^*$ its adjoint. Then

$$\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E)).$$

Other applications to Banach spaces

Remark: Astala and Tylli [AT90, Theorem 4]

There is separable Banach space E and a sequence $(T_n)_n$ of operators $T_n : E \to c_0$ such that

$$\omega(T_n^*(B_{\ell^1})) = 1 \qquad \text{and} \qquad \omega(T_n^{**}(B_E^{**})) \leq w(T_n(B_E)) \leq \frac{1}{n}.$$

Note that this example says, in particular, that there are no constants $m, M > 0$ such that for any bounded operator $T : E \to F$ we have

$$m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E)).$$

Corollary

γ and ω are not equivalent measures of weak noncompactness, namely there is no $N > 0$ such that for any Banach space and any bounded set $H \subset E$ we have

$$\omega(H) \leq N\gamma(H).$$

The results for $C(X)$

If X is a topological space, (Z,d) a metric space and H a relatively compact subset of the space (Z^X,τ_p) we define

$$\operatorname{ck}(H) := \sup_{(h_n)_n \subset H} d\Big(\bigcap_{m\in\mathbb{N}} \overline{\{h_n : n > m\}}^{Z^X}, C(X,Z)\Big).$$

Theorem

Let X be a countably K-determined space, (Z,d) a separable metric space and H a relatively compact subset of the space (Z^X,τ_p). Then, for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_n$ in H such that

$$\sup_{x\in X} d(g(x), f(x)) \overset{(a)}{\leq} 2\operatorname{ck}(H) + 2\hat{d}(H, C(X,Z)) \overset{(b)}{\leq} 4\operatorname{ck}(H)$$

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For the particular case $\operatorname{ck}(H) = 0$ we obtain all known results about compactness in $C_p(X)$ spaces.

Indexes of fragmentability and σ-fragmentability

If X topological space, (Z,d) a metric and $f \in Z^X$ and $\varepsilon > 0$:

Definition

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2. f is $\varepsilon - \sigma$-fragmented by *closed sets* if there is countable family of closed subsets $(X_n)_n$ that covers X such that $f|_{X_n}$ is ε-fragmented for every $n \in \mathbb{N}$.

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Quantitative version of a Rosenthal's result

Theorem

If X is a metric space, E a Banach space and $f \in E^X$ then

$$\frac{1}{2}\,\sigma\text{-frag}_c(f) \leq d(f, B_1(X,E)) \leq \sigma\text{-frag}_c(f).$$

In the particular case $E = \mathbb{R}$ we precisely have

$$d(f, B_1(X)) = \frac{1}{2}\,\sigma\text{-frag}_c(f).$$

Theorem

Let X be a Polish space, E a Banach space and H a τ_p-relatively compact subset of E^X. Then

$$\mathrm{ck}(H) \leq \hat{d}(\overline{H}^{E^X}, B_1(X,E)) \leq 2\,\mathrm{ck}(H).$$

In the particular case when $E = \mathbb{R}$ we have

$$\hat{d}(\overline{H}^{\mathbb{R}^X}, B_1(X)) = \mathrm{ck}(H).$$

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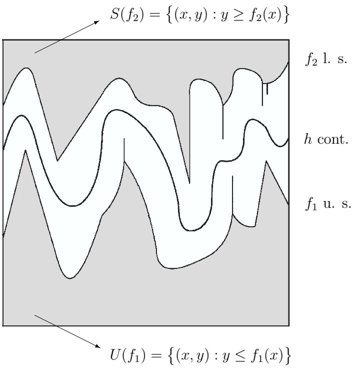
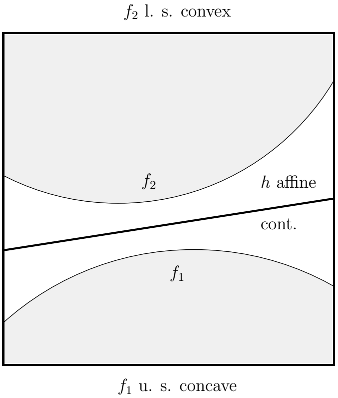


Katetov theorem (X normal)



Hahn-Banach separation theorem